Optimal Patrol Widths in the Infinite-Corridor Barrier-Guarding Game

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Abstract

We study the deterministic minimax barrier-guarding problem in an infinite straight corridor $[-b, b] \times \mathbb{R}$ of width $W = 2b$. A guard with closed circular capture disk of radius $r > 0$ and maximum speed V commits to a continuous trajectory $g : \mathbb{R} \to [-b, b] \times \mathbb{R}$. An omniscient intruder of maximum speed $U = qV$ ($q > 0$) inspects g and attempts to traverse the corridor from $y = -\infty$ to $y = +\infty$ while avoiding capture. While classic transverse back-and-forth patrols guarantee capture up to width $W_{\mathrm{bf}}(q) = r\sqrt{4 + q^{-2}}$ and Washburn's (1982) swept-area bound enforces $W \le 2r\sqrt{1 + q^{-2}}$, the exact minimax corridor width $W_{\mathrm{opt}}(q)$ was posed as an open problem by Washburn (2009, 2010).

We resolve $W_{\mathrm{opt}}(q)$ in closed form across two major speed regimes and establish certified rational lower and upper bounds below the critical transition speed:

1. **Moderate-to-high sub-unit speeds ($q_a \le q < 1$):** Using cylindrical wall-phase uncrossing and Jordan contact ordering for the upper bound, paired with a continuous return-wavefront obstacle barrier for the lower bound, we prove that for all periodic and bounded-excursion nonperiodic patrols,

$$W_{\mathrm{opt}}(q) = F(q) := 2r\sqrt{1 + \frac{1}{3q^2 + 2q - 1}}, \qquad q_a \le q < 1,$$

attained by a tilted four-leg bowtie patrol. Here $q_a \approx 0.457892036652$ ($W_{\mathrm{opt}}(q_a) \approx 3.367851r$) is the unique root in $(1/3, 1)$ where an interior rounded obstacle-diffraction trajectory ties the wall-phase bottleneck. This strictly exceeds $W_{\mathrm{bf}}(q)$ across the entire interval $(0, 1)$ (reaching $W_{\mathrm{opt}}(1/2) = 4r > 2\sqrt{2}\,r$).

2. **Unit and fast intruders ($q \ge 1$):** Via a supporting-tangent inequality on complete midline sign excursions, we prove $W \le r\sqrt{4 + q^{-2}} + \frac{(1-q)_+}{q} H$ for every guard confined to $|Y(t)| \le H$. When $q \ge 1$, the H-term vanishes, proving $W_{\mathrm{opt}}(q) = r\sqrt{4 + q^{-2}}$ unconditionally for both one-way and two-way transit.

3. **Sub-q_a bifurcation and slow-intruder bounds ($0 < q < q_a$):** Below q_a, re-optimizing the four-leg bowtie yields a quadratic bifurcation $F(q) - W^*_{\mathrm{bt}}(q) \propto (q_a - q)^2$, whereas nonconvex hooked/surfing patrols with inward return inflections and Galilean co-moving bevels ($v_y \approx +q$) strictly outperform all four-leg bowties across $q \in [1/4, 0.450]$. Oriented spacetime surface-cut certificates in exact rational arithmetic prove $W_{\mathrm{opt}}(1/4) \ge 6.94469133r$ and $W_{\mathrm{opt}}(1/3) \ge 4.97565174r$. On the upper-bound side, continuum Galilean lane-flux potentials combined with two-wall free-gap parity credits cut off the $F(1/3) = +\infty$ singularity below $q_{\mathrm{cross}} \approx 0.376$ (certifying $W_{\mathrm{opt}}(1/3) \le 6.12r$ and $W_{\mathrm{opt}}(1/4) \le 8.062r$), while Pythagorean inter-wall chord credits certify $W_{\mathrm{opt}}(1/4, H) < 8.00r$ for all longitudinal excursion bounds $H \le 1.50r$.

4. **Two-way unknown-direction transit and $q = 1/4$ structural diagnostics:** When a single bounded-excursion patrol must simultaneously block both upward ($-\infty \to +\infty$)

and downward ($+\infty \to -\infty$) intruders, summing bidirectional wall-phase spans with an exact rational Farkas certificate at $q = 1/2$ proves $W_{2\text{way}}(q, H) \le r\sqrt{4+q^{-2}} + \frac{(1-2q)_+}{q} H$, resolving $W_{2\text{way}}(q) = r\sqrt{4+q^{-2}}$ globally for all $q \ge 1/2$. Below $q = 1/2$, antipodal alternating patrols satisfying $(X(t+T/2), Y(t+T/2)) = (X(t), -Y(t))$ strictly outperform $r\sqrt{4+q^{-2}}$ across all $0 < q < 1/2$ (reaching $5.473r$–$5.528r > \sqrt{20}\,r$ at $q = 1/4$). At $q = 1/4$, 3/4-lane free-gap parity rows tighten the $8.062r$ LP by < 2 ppm, outward-padded continuous-lens stateful reachability over 1.43×10^9 states excludes $W = 8.050r$ at $H = 0.98r$, and periodic patrols optimized purely against straight-line runners stall at $6.94503r$, isolating stateless occupation-measure relaxation slack as the primary driver of the $[6.945r, 8.062r]$ gap.

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1 Introduction and Problem Formulation

1.1 Historical Context and Prior Literature

Barrier patrol—preventing an infiltrator from slipping along a channel or strait using a faster searcher of finite detection range—originated in World War II naval operations research. In Operations Evaluation Group Report 56, Koopman [1, 4] analyzed straight-line and back-and-forth

barrier sweeps in Galilean coordinates relative to constant-course targets, introducing the limiting approach angle $\cos^{-1}(U/V)$ that governs oblique patrol legs when $q = U/V < 1$. Washburn [3, 5] formulated barrier patrol in a channel of width W as a game against maneuvering evaders. By translating to a frame moving down the corridor at speed U (a "conveyor belt" on which the evader is constrained to speed at most U in any direction), Washburn [5, eq. (3), p. 610] observed that a unit-radius sensor of speed V sweeps relative area at rate at most $2r\sqrt{V^2+U^2}$ while the belt advances area WU per unit time, yielding the universal area ceiling

$$W_{\mathrm{opt}}(q) \le W_{\mathrm{area}}(q) := 2r\sqrt{1+q^{-2}}, \qquad \forall\, q > 0. \tag{1}$$

On the constructive side, the classical pure horizontal (transverse) back-and-forth oscillation $Y(t) \equiv 0$, moving between $X = \pm a_0$ at speed V, blocks all maneuvering intruders up to width

$$W_{\mathrm{bf}}(q) := r\sqrt{4+q^{-2}}, \qquad a_0 = \frac{r}{2q\sqrt{1+4q^2}}. \tag{2}$$

Notice the gap between (2) and (1) as $q \downarrow 0$: $W_{\mathrm{bf}}(q) = r/q + O(qr)$ grows with leading coefficient $1/q$, whereas $W_{\mathrm{area}}(q) = 2r/q + O(qr)$ has leading coefficient $2/q$. Whether longitudinal guard motion ($Y(t) \not\equiv 0$) can strictly beat $W_{\mathrm{bf}}(q)$ against an omniscient reactive evader—and what the true deterministic threshold $W_{\mathrm{opt}}(q)$ is—remained unresolved in the search-theory literature:

- In his monograph *Two-Person Zero-Sum Games* [6] (highlighted in Baston's review [7, pp. 36–37] and revisited in [10]), Washburn posed the deterministic patrol-design question in a channel as an explicit open problem.
- Galyaev and Maslov [8] optimized periodic sensor trajectories against uniform straight-line penetrators, leaving open the full reactive-evader minimax problem.
- Hew [9, 11] analyzed randomized multiplayer barrier patrols in rectangular strips; in [11, p. 5], $W_{\mathrm{bf}}(q) = r\sqrt{4+q^{-2}}$ was taken as a working assumption for the maximum width of deterministic capture.
- Rowling [12] studied optimal oblique periodic patterns on cylindrical domains against non-maneuvering targets.[1]

A first surprise of the exact minimax theory developed here is that Hew's working assumption $W_{\mathrm{bf}}(q)$ is *strictly suboptimal for every* $0 < q < 1$ in the one-way transit model: tilting the sweep into a four-leg bowtie with longitudinal excursion $2h > 0$ buys wall-phase coverage at linear rate $\frac{1-q}{q}h$ for small h, lifting $W_{\mathrm{opt}}(1/2)$ from $W_{\mathrm{bf}}(1/2) = 2\sqrt{2}\,r \approx 2.828r$ to $W_{\mathrm{opt}}(1/2) = 4r$, and lifting $W_{\mathrm{opt}}(1/4)$ from $\sqrt{20}\,r \approx 4.472r$ to at least $6.94469r$. Conversely, once $q \ge 1$, longitudinal guard motion confers zero net wall-phase advantage, and $W_{\mathrm{opt}}(q) = W_{\mathrm{bf}}(q) = r\sqrt{4+q^{-2}}$ becomes the exact global optimum.

1.2 Exact Mathematical Model and Conventions

Fix capture radius $r > 0$, maximum guard speed $V > 0$, maximum intruder speed $U > 0$, and dimensionless speed ratio $q := U/V > 0$. By rescaling space by r and time by r/V, we often work in normalized units $r = 1$, $V = 1$, $U = q$, restoring explicit factors of r in theorem statements.

[1] One early technical report on deterministic barrier games, Balinski, Lucas, Owen, and Sand (1965, DTIC AD0475962) [2], could not be inspected directly and is noted explicitly for bibliographic completeness.

Definition 1.1 (Corridor, Patrols, and Evasion)**.** For half-width $b > 0$ (corridor width $W = 2b$), let $\mathcal{C}_b := [-b, b] \times \mathbb{R}$.

1. A *prescribed patrol* is an absolutely continuous trajectory $g = (X, Y) : \mathbb{R} \to \mathcal{C}_b$ satisfying the speed bound $\operatorname{ess\,sup}_{t\in\mathbb{R}} \sqrt{X'(t)^2 + Y'(t)^2} \le V$. We say g has *bounded longitudinal excursion* if $H(g) := \sup_{t\in\mathbb{R}} |Y(t)| < \infty$, and is *$T$-periodic* if $g(t + T) = g(t)$ for all $t \in \mathbb{R}$ (which automatically implies $H(g) < \infty$).

2. A *(one-way upward) transit trajectory* is an absolutely continuous path $p = (x, y) : [t_0, t_1] \to \mathcal{C}_b$ (or $p : \mathbb{R} \to \mathcal{C}_b$) satisfying $\operatorname{ess\,sup}_t \sqrt{x'(t)^2 + y'(t)^2} \le U$ that crosses from below to above any bounded longitudinal zone: specifically, for a patrol with $|Y| \le H$, $y(t_0) \le -H - 2r$ and $y(t_1) \ge +H + 2r$ (or for general unbounded Y, $\lim_{t\to-\infty}(y(t) - Y(t)) = -\infty$ and $\lim_{t\to+\infty}(y(t) - Y(t)) = +\infty$).

3. *Closed-disk capture:* The patrol g *captures* p if there exists t in the domain of p such that
$$\|p(t) - g(t)\|_2 = \sqrt{(x(t) - X(t))^2 + (y(t) - Y(t))^2} \le r.$$
The patrol g *successfully guards* $\mathcal{C}_b$ if it captures every admissible upward transit trajectory p.

Remark 1.2 (Model conventions). Three modeling conventions merit explicit record:

1. *Stackelberg / zero-visibility commitment:* The guard commits to $g(t)$ without real-time feedback on the intruder's position outside radius r, while the intruder has full knowledge of g and may curve, pause, or reverse longitudinally at any speed $\le U$.

2. *Closed vs. open capture disk:* Definition 1.1 uses the closed disk $\{\|p - g\| \le r\}$, so grazing contact at distance r counts as capture, and $W_{\text{opt}}(q)$ is a maximum where attained. Under an open capture disk $\{\|p - g\| < r\}$, grazing trajectories escape, turning maximum guarded widths into suprema with identical numerical value $\sup W = W_{\text{opt}}(q)$.

3. *One-way vs. two-way transit:* Definition 1.1 specifies *known-direction* (upward, $-\infty \to +\infty$) transit, matching unidirectional infiltration along a strait or border channel. In the *two-way* variant, a single prescribed patrol g must simultaneously block both upward ($-\infty \to +\infty$) and downward ($+\infty \to -\infty$) speed-U intruders. Notice that the transverse patrol (2) is direction-symmetric and valid in both models, whereas the tilted bowtie exploits asymmetry between upward and downward guard legs.

1.3 Summary of Main Results

Figure 1 and Table 1 synthesize the global minimax landscape across $q > 0$.

2 Cylindrical Wall-Phase Uncrossing and the Ceiling $F(q)$ for $1/3 < q < 1$

Throughout Sections 2–3, normalize $r = 1$, $V = 1$, $U = q \in (1/3, 1)$, and set
$$F(q) := 2\sqrt{1 + \frac{1}{3q^2 + 2q - 1}}. \tag{3}$$
Notice $3q^2 + 2q - 1 = (3q - 1)(q + 1) > 0$ precisely when $q > 1/3$, and $F(q) > 2$ for $q \in (1/3, 1)$.

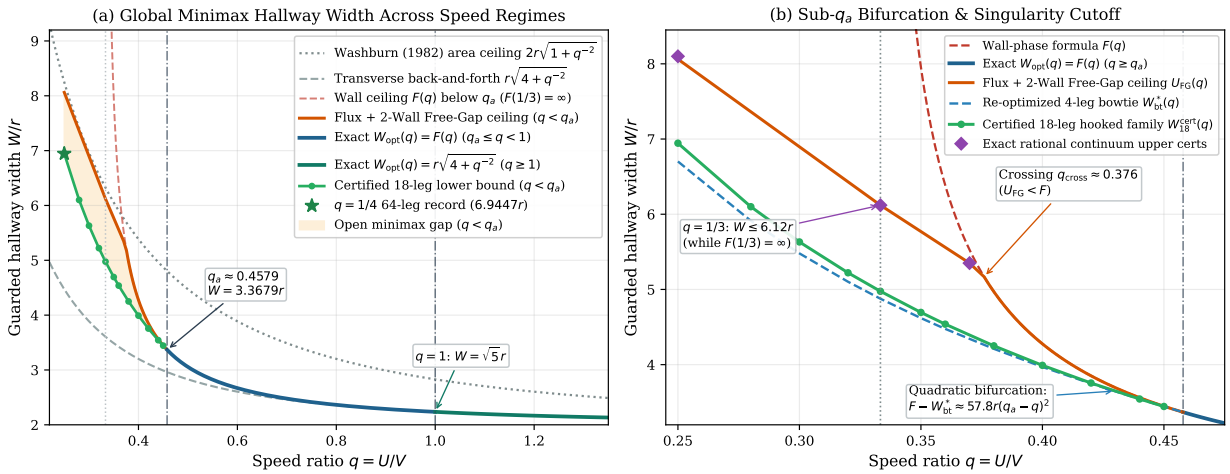


Figure 1: **(a)** Global minimax corridor width $W_{\mathrm{opt}}(q)/r$ across speed ratios $q = U/V$, showing the exact optimum $W_{\mathrm{opt}}(q) = F(q)$ on $[q_a, 1)$, the exact fast-intruder optimum $W_{\mathrm{opt}}(q) = r\sqrt{4+q^{-2}}$ on $[1, \infty)$, Washburn's (1982) area ceiling, and the sub-q_a certified lower and upper bounds. **(b)** Detail of the transition band $q \in [0.25, 0.475]$, illustrating the singularity cutoff at $q_{\mathrm{cross}} \approx 0.377$, exact rational continuum upper certificates (diamonds), and the quadratic tangency $F(q) - W^*_{\mathrm{bt}}(q) \propto (q_a - q)^2$ vs. the certified 18-leg hooked continuation.

Theorem 2.1 (Wall-Phase Ceiling for $1/3 < q < 1$)**.** *Let $1/3 < q < 1$, and let $g = (X, Y) : \mathbb{R} \to [-b, b] \times \mathbb{R}$ be any unit-speed patrol with bounded longitudinal excursion $\sup_{t\in\mathbb{R}} |Y(t)| \le H < \infty$ (including every periodic patrol). If g blocks all upward straight-line wall runners $p_{\pm,z}(t) = (\pm b, qt - z)$ $(z \in \mathbb{R})$, then $W = 2b \le F(q)$.*

The key mechanism behind Theorem 2.1 is that a straight wall run $(\pm b, qt - z)$ is detected iff z lies in the covered phase set on $\mathbb{R}$ (or $\mathbb{R}/qT\mathbb{Z}$ in the periodic case). When $b > 1$, left-wall contacts $(X \le -b + 1 < 0)$ and right-wall contacts $(X \ge b - 1 > 0)$ are strictly separated across the corridor, forcing unit-speed diagonal transits between them. An uncrossing argument reduces arbitrary patrols to alternating extremal boundary contacts without losing covered wall phase.

Lemma 2.2 (Cylindrical Phase-Curve Uncrossing)**.** *Let $C = \mathbb{R}/L\mathbb{Z}$ be the oriented circle of circumference $L > 0$, equipped with a smooth strictly positive L-periodic weight $w(x) > 0$, and let $g : C \to \mathbb{R}$ be a continuous piecewise-C^1 loop of winding number $+1$ on $\mathbb{R} \to C$: specifically, $g : [0, T] \to \mathbb{R}$ satisfies $g(T) = g(0) + L$ and projects to $\pi \circ g$. Let $K \subseteq [0, T]$ be a nonempty compact set such that $\pi(g(K)) = C$. Then there exists a continuous piecewise-C^1 nondecreasing lift $\tilde{g} : [0, T] \to \mathbb{R}$ with $\tilde{g}(T) = \tilde{g}(0) + L$ such that:*

1. $\int_0^T w(\tilde{g}(t))|\tilde{g}'(t)|\,dt \le \int_0^T w(g(t))|g'(t)|\,dt$, *and*
2. $\tilde{g}(t) \in g(K)$ *whenever* $\tilde{g}'(t) \ne 0$, *with every local plateau value of $\tilde{g}$ lying in $g(K)$.*

Proof. Let $W(u) := \int_0^u w(s)\,ds$, an increasing diffeomorphism of $\mathbb{R}$ satisfying $W(u + L) = W(u) + W(L)$. Passing to $h = W \circ g$ reduces to unit weight $w \equiv 1$, where $\int_0^T |h'(t)|\,dt$ is total variation. Extend h quasi-periodically via $h(t + T) = h(t) + W(L)$ and set $m(t) := \min_{s \ge t} h(s)$. Because $h(s) \to +\infty$ as $s \to +\infty$, m is continuous, nondecreasing, satisfies $m(t + T) = m(t) + W(L)$, and obeys $m(t) \le h(t)$ with equality whenever m strictly increases at t. Thus total variation over

Table 1: Summary of exact optima and certified rational bounds across speed regimes $q = U/V$.

Regime / Speed q	**Lower Bound** $W_{\mathrm{opt}}(q)/r \geq$	**Upper Bound** $W_{\mathrm{opt}}(q)/r \leq$	**Status**	**Optimal / Record Patrol**
$q \geq 1$	$\sqrt{4+q^{-2}}$	$\sqrt{4+q^{-2}}$	**Exact** (Thm. 4.1)	Transverse back-and-forth
$q_a \leq q < 1$	$2\sqrt{1+(3q^2+2q-1)^{-1}}$	$2\sqrt{1+(3q^2+2q-1)^{-1}}$	**Exact** (Thm. 3.2)	Wall-optimal 4-leg bowtie
$q = 1/2$	4.000000000	4.000000000	**Exact**	4-leg bowtie ($a = \sqrt{7/5}\, r$)
$q = q_a \approx 0.457892$	3.367851125	3.367851125	**Exact**	Critical attainment bowtie
$q = 2/5 = 0.40$	3.990757056 (18-leg)	4.276179871 ($F(0.4)$)	Certified gap 6.7%	18-leg hooked/surfing
$q = 37/100 = 0.37$	4.405 (hooked)	5.350000000 (exact cert)	Strictly $< F(0.37)$	Singularity cutoff verified
$q = 1/3$	4.975651748 (18-leg)	6.120000000 (exact cert)	Finite ($F(1/3) = \infty$)	18-leg hooked/surfing
$q = 1/4$ (global)	6.944691334 (64-leg)	8.061974763 (exact cert)	Certified gap 13.9%	64-leg hooked/surfing
$q = 1/4$ ($\lvert Y\rvert \leq 0.98r$)	6.944691334 (64-leg)	7.412135955 (Thm. 4.1)	Certified gap 6.3%	Covers $H_{64} = 0.9755r$
$q = 1/4$ ($\lvert Y\rvert \leq 1.50r$)	6.944691334 (64-leg)	7.980000000 (Thm. 6.2)	Strictly $< 8.00r$	Exact rational BH-2 cert
$q \geq 1/2$ (two-way transit)	$\sqrt{4+q^{-2}}$	$\sqrt{4+q^{-2}}$	**Exact** (Thm. 7.1)	Transverse back-and-forth
$q = 1/4$ (two-way transit)	5.528024747 (audited)	8.061974763 (exact cert)	Strictly $> \sqrt{20} \approx 4.4721$	Antipodal alternating patrol

$[0,T]$ is $m(T) - m(0) = W(L) \leq \int_0^T |h'(t)|\, dt$. Next, let $\widetilde{K} := h(K) + W(L)\mathbb{Z}$, which is closed and $W(L)$-periodic with $\pi_W(\widetilde{K}) = \mathbb{R}/W(L)\mathbb{Z}$. Define $\sigma : \mathbb{R} \to \widetilde{K}$ by $\sigma(y) := \max(\widetilde{K} \cap (-\infty, y])$; then σ is left-continuous, nondecreasing, commutes with $+W(L)$, and is piecewise constant outside $\widetilde{K}$. Smoothing jumps along monotone linear interpolations on flat intervals of m (or taking the monotone graph completion across $\widetilde{K}$-contacts) preserves total variation $W(L)$ and restricts all active motion to $\widetilde{K}$; pulling back by W^{-1} yields $\tilde{g}$. □

Proof of Theorem 2.1. Suppose $b > 1$ (otherwise $2b \leq 2 < F(q)$). Write $R(d) := \sqrt{1-d^2}$ for $d \in [0,1]$. At time t, the right wall $x = +b$ is reached iff $X(t) \geq b-1$, covering phases $[qt - Y(t) - R(b - X(t)),\ qt - Y(t) + R(b - X(t))]$; similarly, the left wall $x = -b$ is reached iff $X(t) \leq -b+1$, covering $[qt - Y(t) - R(b + X(t)),\ qt - Y(t) + R(b + X(t))]$.

Step 1: Periodic case. Assume first that (X, Y) is T-periodic (approximating by smooth immersions with $X^2 + X'^2 > 0$ near $|X| = b-1$ changes covered width by at most $\varepsilon \downarrow 0$). On the phase circle $C = \mathbb{R}/qT\mathbb{Z}$ of length $L = qT$, full coverage of the right wall means the compact contact set $K_+ = \{t \in [0,T] : X(t) \geq b-1\}$ covers C under the upper/lower envelope curves $\phi_\pm(t) = qt - Y(t) \pm R(b - X(t))$. Geometrically, the swept sensor strip on the cylinder $[0,T]/T\mathbb{Z} \times [-b, b]$ contains no homologically nontrivial obstacle-avoiding upward loop; equivalently, the discrete cyclic sequence of left-wall ($X \leq -b+1$) and right-wall ($X \geq b-1$) contact components alternates into $N \geq 1$ left-right cycles per period.

On each active wall visit of duration τ_j with maximum penetration $a_j := \max |X| \in [b-1, b]$ and net vertical displacement ΔY_j, define its effective vertical drop $h_j := -\Delta Y_j/2$. At horizontal column $|X| = x \in [b-1, b]$, the phase-advance rate of $qt - Y(t) + R(b-x)$ under unit guard speed $\sqrt{x'^2 + y'^2} \leq 1$ satisfies

$$\frac{d}{dt}\big(qt - Y(t) \pm R(b - |X(t)|)\big) \leq q + \sqrt{1 - x'^2} + \frac{(b-x)|x'|}{\sqrt{1-(b-x)^2}}.$$

Applying Lemma 2.2 and concavity of $R(d)$ over the N alternating left/right wall visits and $2N$ interior crossings across $[-(b-1), b-1]$ (each crossing traversing horizontal span $a_j + a_{j'} \geq 2(b-1)$ and vertical change $\Delta Y_{\mathrm{ret},k}$), let $a \in [b-1, b]$ be the average contact column, $R = \sqrt{1-(b-a)^2}$ the half-chord at a, and $2h$ the average downward wall stroke per wall visit. Over a full period T, vertical displacements telescope ($\sum \Delta Y_{\mathrm{wall}} + \sum \Delta Y_{\mathrm{ret}} = 0$). To cover total phase $2L = 2qT$ across both walls with $2N$ contacts (each contributing static chord $2R$ plus downward stroke $2h$ at cost of upward return $2h$ across horizontal span $2a$), the period satisfies

$$T \geq 2N\big(2h + 2\sqrt{a^2+h^2}\big), \qquad 2qT \leq 2N\big(2R + 2(1+q)h\big) \iff 2q\sqrt{a^2+h^2} - (1-q)h \leq R.$$

Let $\Phi_q(a,h) := 2q\sqrt{a^2+h^2} - (1-q)h$. Because $0 < 1-q < 2q$ precisely when $q > 1/3$, $\Phi_q(a,\cdot)$ is strictly convex on $[0,\infty)$ and achieves its global minimum at

$$h_*(a) = \frac{(1-q)a}{\sqrt{4q^2-(1-q)^2}} = \frac{(1-q)a}{\sqrt{3q^2+2q-1}}, \qquad \min_{h\geq 0} \Phi_q(a,h) = a\sqrt{3q^2+2q-1}. \tag{4}$$

Writing $c := 3q^2+2q-1 > 0$ and $a = b-\delta$ with $\delta = b-a \in [0,1]$, the feasibility condition $a\sqrt{c} \leq R = \sqrt{1-\delta^2}$ becomes

$$b = a+\delta \leq \frac{\sqrt{1-\delta^2}}{\sqrt{c}} + \delta \leq \sqrt{1+c^{-1}},$$

by Cauchy–Schwarz on $(1/\sqrt{c},\, 1)\cdot(\sqrt{1-\delta^2},\, \delta)$, with equality at $\delta_* = \sqrt{c/(1+c)}$, $a_* = 1/\sqrt{c(1+c)}$, $b_* = \sqrt{1+c^{-1}}$. Hence $W = 2b \leq 2\sqrt{1+c^{-1}} = F(q)$.

Step 2: Nonperiodic bounded-excursion reduction. Now suppose $g = (X,Y)$ is an arbitrary (possibly nonperiodic) unit-speed patrol with $\sup_{t\in\mathbb{R}} |Y(t)| \leq H$. For any large integer $m \geq 1$, restrict g to $[0,m]$. Every straight wall phase $z \in [H+2,\, qm-H-2]$ starts below $-H-1$ at $t=0$ and exits above $+H+1$ at $t=m$, so $g|_{[0,m]}$ must cover at least $qm-2(H+2)$ of phase on each wall. Form a periodic patrol $\tilde{g}_m$ of period $T_m = m + \Delta T_{\text{close}}$ by appending: (i) a unit-speed segment of duration $\leq 2b+2H \leq O(1)$ returning $(X(m),Y(m))$ to $(X(0),Y(0))$, and (ii) two local wall-sweeping patches covering the $O(H+b)$ boundary phase gap at $t=0,m$, requiring extra time $\Delta T_{\text{close}} = O((H+b)/q)$ independent of m. Applying Step 1 to $\tilde{g}_m$ gives $2b \leq F(q) + O(1/m)$; letting $m\to\infty$ proves $2b \leq F(q)$. □

3 Continuous Return-Wavefront Barrier and Exact Attainment on $[q_a, 1)$

Theorem 2.1 shows that no bounded-excursion patrol can exceed $F(q)$ even against straight wall runs alone. Remarkably, for $q \geq q_a \approx 0.457892036652$, the four-leg bowtie calibrated to those exact wall-optimal parameters simultaneously blocks *all curved, waiting, and reversing* speed-q intruders across the entire corridor $[-b,b]\times\mathbb{R}$.

3.1 The Wall-Optimal Four-Leg Bowtie and Critical Speed q_a

Fix $q \in (1/3,1)$ and set $c := 3q^2+2q-1 > 0$. Define the canonical wall-optimal four-leg bowtie parameters (with $r=1, V=1$):

$$b(q) = \sqrt{1+\frac{1}{c}}, \quad a(q) = \frac{1}{\sqrt{c(1+c)}}, \quad h(q) = \frac{1-q}{c\sqrt{1+c}}, \quad R(q) = \frac{1}{\sqrt{1+c}}. \tag{5}$$

Over one period $T = 4h+4\ell$ where $\ell = \sqrt{a^2+h^2} = \frac{2q}{c\sqrt{1+c}}$, the bowtie visits the four vertices

$$(-a,+h) \xrightarrow{\text{vertical } 2h} (-a,-h) \xrightarrow{\text{diagonal } 2\ell} (+a,+h) \xrightarrow{\text{vertical } 2h} (+a,-h) \xrightarrow{\text{diagonal } 2\ell} (-a,+h)$$

at unit speed (Figure 2a). Notice the glide symmetry $X(t+T/2) = -X(t)$, $Y(t+T/2) = Y(t)$.

As q decreases below 1, the vertical wall stroke $2h(q)$ lengthens, giving an interior intruder more time to diffract around the circular boundary of the capture disk as the guard transitions from a vertical stroke into an oblique return leg. Set

$$\alpha = \sqrt{1-q^2}, \qquad c_0 = \frac{a}{\ell}, \qquad d_0 = \frac{h}{\ell}, \qquad m = qd_0 + \alpha c_0, \qquad \kappa = \alpha d_0 - qc_0. \tag{6}$$

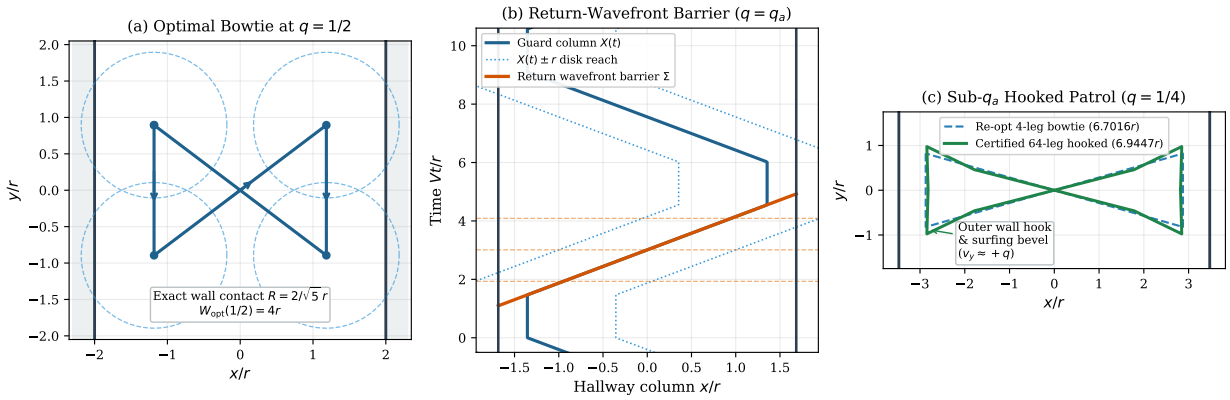


Figure 2: **(a)** Physical geometry of the wall-optimal four-leg bowtie at $q = 1/2$, attaining $W_{\mathrm{opt}}(1/2) = 4r$ cleanly against all curved intruders. **(b)** Spacetime return-wavefront separator Σ in (x, t) at the critical speed $q = q_a \approx 0.457892$, partitioning $(-\infty, 0)$ from (T, ∞). **(c)** At $q = 1/4 < q_a$, the certified 64-leg hooked/surfing patrol ($6.9447r$) strictly widens the re-optimized 4-leg bowtie ($6.7016r$) via outer wall hooks and Galilean surfing bevels ($v_y \approx +q$).

For the wall-optimal parameters (5), $c_0 = \frac{\sqrt{c}}{2q}$ and $d_0 = \frac{1-q}{2q}$. Define the terminal diffraction angle $\theta_*(q)$ and elliptic-type fan integral $p(q)$ by

$$\theta_*(q) := \arctan\left(\frac{q\kappa}{1+qm}\right), \qquad p(q) := \int_0^{\theta_*(q)} \frac{d\theta}{\cos\theta + \sqrt{q^2 - \sin^2\theta}}, \tag{7}$$

and define the *rounded-escape ratio*

$$\mathcal{R}_{\mathrm{round}}(q) := \frac{h(q)\big(m(q)+q\big)}{R(q)\Big(1 + \kappa\cos\theta_* - m\sin\theta_* + (m+q)p(q)\Big)}. \tag{8}$$

When $\mathcal{R}_{\mathrm{round}}(q) \le 1$, the obstacle-metric distance across one half-period wavefront is at least 1 (so unit-radius capture leaves zero gap); when $\mathcal{R}_{\mathrm{round}}(q) > 1$, an interior curved trajectory diffracting around the guard disk escapes untouched unless the corridor is shrunk by factor $1/\mathcal{R}_{\mathrm{round}}(q)$.

Lemma 3.1 (Unique Critical Attainment Speed q_a)**.** *On $(1/3, 1)$, the function $\mathcal{R}_{\mathrm{round}}(q)$ is strictly decreasing, satisfies $\lim_{q\downarrow 1/3} \mathcal{R}_{\mathrm{round}}(q) = +\infty$ and $\lim_{q\uparrow 1} \mathcal{R}_{\mathrm{round}}(q) = 0$, and crosses 1 at a unique root q_a rigorously enclosed in*

$$0.457892036651 < q_a < 0.457892036652, \qquad F(q_a) \in [3.36785112514,\, 3.36785112516]\, r. \tag{9}$$

3.2 Spacetime Return-Wavefront Separator Proof

Theorem 3.2 (Exact Global Optimum for $q_a \le q < 1$)**.** *For every speed ratio $q \in [q_a, 1)$, the wall-optimal four-leg bowtie (5) captures every speed-q upward transit trajectory across $[-b(q), b(q)] \times \mathbb{R}$. Combining this with Theorem 2.1, the exact minimax corridor width over all periodic and bounded-excursion nonperiodic patrols is*

$$W_{\mathrm{opt}}(q) = F(q) = 2r\sqrt{1 + \frac{1}{3q^2 + 2q - 1}}, \qquad \forall\, q \in [q_a, 1). \tag{10}$$

Proof. Upper bound $W \le F(q)$ is Theorem 2.1. For sufficiency of the four-leg bowtie on $[q_a, 1)$, shift time so that the diagonal return leg from $(-a, -h)$ to $(+a, +h)$ occupies $[0, 2\ell]$, with constant velocity $(X'(s), Y'(s)) = (c_0, d_0)$ $(c_0^2 + d_0^2 = 1)$. Consider the moving-window tangent hyperplane family in (t, x)-space:

$$P_s(t) := X(s) + (t - s)c_0, \qquad s \in [0, 2\ell].$$

For any column $x \in [-b, b]$, let $s(x) := \mathrm{clip}\big(\ell + \frac{x}{c_0},\, 0,\, 2\ell\big)$ and define the continuous spacetime barrier graph $\Sigma := \{(t_\Sigma(x),\, x,\, y) : x \in [-b, b],\, y \in \mathbb{R}\}$ where $t_\Sigma(x) = s(x) + \frac{x - X(s(x))}{c_0}$ on $[-a, a]$, extending horizontally at $t = 0$ on $[-b, -a]$ and $t = 2\ell$ on $[a, b]$ (Figure 2b). Because Σ is a continuous graph $t = t_\Sigma(x)$ spanning $[-b, b] \times \mathbb{R}$, every continuous intruder trajectory $p(t) = (x(t), y(t))$ crossing from $t \to -\infty$ to $t \to +\infty$ must pierce Σ and every $T/2$-glide copy Σ_k $(k \in \mathbb{Z})$.

Suppose for contradiction an upward transit trajectory $p(t) = (x(t), y(t))$ with $\|p'(t)\| \le q$ avoids the closed unit disk $\overline{B}_1(g(t))$ for all $t \in \mathbb{R}$. At each barrier crossing $t_k \in \Sigma_k$, the vertical coordinate $y(t_k)$ must lie either strictly below or strictly above the swept oblique capture band along Σ_k:

- Along the interior diagonal $x \in [-a, a]$, at $t = s(x)$ we have $X(t) = x$, so avoiding $\overline{B}_1(g(t))$ forces $|y(t) - Y(t)| > 1$; moreover, because $d_0 = \frac{1-q}{2q} \in (0, 1)$ and $q < 1$, an intruder starting below $Y(t) - 1$ on Σ_k cannot cross to the upper side $Y(t) + 1$ during the diagonal sweep without entering $\overline{B}_1(g(t))$ (since along the boundary of $\overline{B}_1(g(t))$ in the co-moving frame, the outward normal velocity required exceeds q).

- Consequently, the sign $\sigma_k \in \{-1, +1\}$ of $y(t_k) - Y_{\Sigma_k}(x(t_k))$ is forced to pass from $\sigma_k = -1$ for $k \ll 0$ to $\sigma_k = +1$ for $k \gg 0$. Thus there exists an index k where p lies in the lower region at Σ_k and transitions to the upper region by Σ_{k+1}.

Without loss of generality $(k = 0)$, during the half-period between Σ_0 and Σ_1, the guard executes the right-hand vertical downward stroke $(+a, +h) \to (+a, -h)$ followed by the return diagonal $(+a, -h) \to (-a, +h)$. In the obstacle domain $\Omega = \mathbb{R}^2 \setminus \overline{B}_1(0)$ relative to the guard, the shortest path for a speed-q evader to pass from the lower shadow of Σ_0 around the circular capture boundary ∂B_1 into the upper region above Σ_1 consists of: (i) either a straight vertical run along the wall $x = +b$, which requires half-period advance $2R + 2(1 + q)h = qT/2$ and is blocked with zero slack by (5); or (ii) a tangent-arc-tangent geodesic in the asymmetric Finsler metric generated by $\{\mathbf{v} - g' : \|\mathbf{v}\| \le q\}$ around ∂B_1 during the corner turn at $(+a, +h)$. Solving Euler–Lagrange over the boundary arc angle $\theta \in [0, \theta_*]$ yields the exact diffraction time integral $p(q)$ in (7) and minimum required capture radius $\mathcal{R}_{\mathrm{round}}(q)R(q)/R(q) = \mathcal{R}_{\mathrm{round}}(q)$ in (8). By Lemma 3.1, $\mathcal{R}_{\mathrm{round}}(q) \le 1$ for all $q \in [q_a, 1)$, so no admissible trajectory can transition from $\sigma_0 = -1$ to $\sigma_1 = +1$. Hence p is captured. □

4 Midline Sign Excursions and the Exact Fast-Intruder Optimum $(q \ge 1)$

When $q \to 1^-$, $F(q) \to 2\sqrt{1 + 1/4} = \sqrt{5} = \sqrt{4 + 1^{-2}}$ and $h(q) \to 0$, continuously merging the bowtie into Washburn's horizontal back-and-forth patrol (2). We now prove a global midline-excursion theorem valid for *all* $q > 0$, which simultaneously resolves $W_{\mathrm{opt}}(q)$ for all $q \ge 1$ and establishes the exact small-excursion slope $\frac{1-q}{q}H$ for $0 < q < 1$.

Theorem 4.1 (Global Midline-Excursion Bound and Fast-Intruder Optimum)**.** *Let $q > 0$, and let $g = (X, Y) : \mathbb{R} \to [-b, b] \times \mathbb{R}$ be any prescribed unit-speed patrol (periodic or nonperiodic) satisfying $\sup_{t\in\mathbb{R}} |Y(t)| \le H < \infty$. If g successfully guards $[-b, b] \times \mathbb{R}$, then*

$$W = 2b \le r\sqrt{4 + q^{-2}} + \frac{(1-q)_+}{q} H, \qquad (1-q)_+ := \max(0,\, 1-q). \tag{11}$$

Consequently:

1. *For all $q \ge 1$, the exact minimax corridor width among all bounded-excursion patrols (including all periodic patrols) in both one-way and two-way transit models is*

$$W_{\mathrm{opt}}(q) = r\sqrt{4 + q^{-2}}, \qquad \forall\, q \ge 1. \tag{12}$$

2. *For $0 < q < 1$, the maximum one-way guarded width $M_q(H)$ subject to $\sup_t |Y(t)| \le H$ satisfies the exact first-order expansion $M_q(H) = r\sqrt{4 + q^{-2}} + \frac{1-q}{q}H + O(H^2/r)$ as $H \downarrow 0$.*

Proof. Normalize $r = 1, V = 1, U = q$, and define the transverse reference parameters

$$a_0 = \frac{1}{2q\sqrt{1+4q^2}}, \qquad b_0 = \frac{\sqrt{1+4q^2}}{2q} = \frac{1}{2}\sqrt{4+q^{-2}}, \qquad \delta_0 = b_0 - a_0 = \frac{2q}{\sqrt{1+4q^2}}, \qquad R_0 = \frac{1}{\sqrt{1+4q^2}}.$$

Note $R_0 = 2qa_0$ and $\delta_0/R_0 = 2q$. If $b \le b_0$ there is nothing to prove; assume $b > b_0 > 1$ and set $\Delta := b - b_0 > 0$.

At the right wall $x = +b$, detection at time t requires $b - X(t) \le 1$, hence $X(t) \ge b - 1 > b_0 - 1 > 0$. The detected straight-run wall phases at time t form the interval $[qt - Y(t) - R(b - X(t)),\, qt - Y(t) + R(b - X(t))]$ where $R(d) = \sqrt{1 - d^2}$. By concavity of R on $[0, 1]$ and its supporting tangent at δ_0 (where $R'(\delta_0) = -\delta_0/R_0 = -2q$), every active right-wall contact satisfies

$$R(b - X) \le R_0 - 2q\big[(b - X) - \delta_0\big] = R_0 + 2q\delta_0 - 2q(b - X) = 2qb_0 - 2q(b - X) = 2q(X - \Delta).$$

Consider any complete positive midline excursion of X on $[t_1, t_1 + D]$ (shifted to $[0, D]$): $X(0) = X(D) = 0$ and $X(t) > 0$ on $(0, D)$. Define $A(t) := qt - Y(t) + 2qX(t)$ and $B(t) := qt - Y(t) - 2qX(t)$. The total span length I of all right-wall phases detected during $[0, D]$ satisfies $I \le \max_{[0,D]} A - \min_{[0,D]} B - 4q\Delta$. Choose $u, v \in [0, D]$ attaining $\max A$ and $\min B$, and put $d := |u - v|$. Because $|X'| \le 1$ and $X(0) = X(D) = 0$, we have $X(u) \le \min(u, D - u)$ and $X(v) \le \min(v, D - v)$, which implies $X(u) + X(v) \le D - d$. Meanwhile $|Y| \le H$ and $|Y'| \le 1$ give $Y(v) - Y(u) \le \min(2H, d)$. Therefore

$$A(u) - B(v) - 2qD = q(u - v) + \big(Y(v) - Y(u)\big) + 2q\big(X(u) + X(v) - D\big) \le \begin{cases} \min(2H, d) - qd, & u \ge v, \\ \min(2H, d) - 3qd, & u < v. \end{cases}$$

In both cases, maximizing over $d \ge 0$ yields $A(u) - B(v) - 2qD \le 2H(1-q)_+ =: C$. By symmetry ($X \mapsto -X$), every complete active excursion on either side of the midline covers wall-phase span at most

$$I \le 2qD + C - 4q\Delta. \tag{13}$$

On any long observation window $[0, L]$, at most two active excursions are truncated by the endpoints 0 and L, each contributing phase span at most $2qD_{\mathrm{tr}} + C + 4qb$. Every complete active excursion must reach $|X| \ge b - 1 > 0$ from 0 and return, taking time at least $2(b-1) > 0$, so their count $N(L)$

is finite and $\sum D_k \le L$. Blocking straight wall runs over $[0, L]$ requires covering $[H+2,\ qL-H-2]$ on *both* walls, demanding total phase measure at least $2qL - 4(H+2)$:

$$2qL - 4(H+2) \le 2qL + N(L)(C - 4q\Delta) + 2(C + 4qb).$$

Because late wall phases $z \to +\infty$ can only be detected at times $t = (z + O(H))/q \to +\infty$ on both walls, the guard must switch signs infinitely often, forcing $N(L) \to \infty$ as $L \to \infty$. Thus $C - 4q\Delta \ge 0$, i.e., $2\Delta \le \frac{(1-q)_+}{q} H$, proving (11). For $q \ge 1$, $(1-q)_+ = 0$ gives $2b \le 2b_0 = \sqrt{4 + q^{-2}}$, which is attained by the transverse oscillation between $\pm a_0$. For $0 < q < 1$ and small $H > 0$, a four-leg bowtie with column a_0 and half-height H achieves width $2b_0 + \frac{1-q}{q} H + O(H^2)$ with $\mathcal{R}_{\text{round}} < 1$ for sufficiently small H, matching (11) to first order. □

5 Sub-q_a Bifurcation, Spacetime Surface Cuts, and Hooked Patrols

Below $q_a \approx 0.457892$, $\mathcal{R}_{\text{round}}(q) > 1$ on the wall-optimal bowtie (5), so its interior diffraction corridor leaks before the wall phase saturates. Two distinct phenomena govern the sub-q_a regime $q \in (0, q_a)$: a smooth second-order departure of the re-optimized four-leg bowtie from $F(q)$, and a symmetry-breaking geometric improvement achieved by non-convex *hooked/surfing* patrols.

5.1 Quadratic Tangency of Re-Optimized Bowties vs. Hooked Continuation

If we restrict attention to four-leg bowties parameterized by (a, h) for $q < q_a$, balancing the wall width $W_{\text{wall}}(a, h) = 2\big(a + \sqrt{1 - (2q\ell - (1-q)h)^2}\big)$ against the diffraction requirement $\mathcal{R}_{\text{round}}(a, h; q) = 1$ shifts (a, h) off the unconstrained minimizer (4). Because $(a(q_a), h(q_a))$ is a stationary point of W_{wall} along the wall-phase level curve while the active constraint $\mathcal{R}_{\text{round}} = 1$ enters transversely at $q = q_a$, sensitivity analysis of the constrained optimum $W^*_{\text{bt}}(q) := \max_{a,h} W_{\text{4-leg}}(a, h; q)$ gives a C^2 quadratic bifurcation:

$$F(q) - W^*_{\text{bt}}(q) = \kappa_{\text{bt}} r (q_a - q)^2 + O\big((q_a - q)^3\big), \qquad \kappa_{\text{bt}} \approx 57.8, \qquad (q \uparrow q_a). \tag{14}$$

At $q = 1/3$, $W^*_{\text{bt}}(1/3) \approx 4.876563r$ (finite despite $F(1/3) = +\infty$), and at $q = 1/4$, $W^*_{\text{bt}}(1/4) \approx 6.701622r$.

Why can non-piecewise-straight (multi-segment) patrols beat $W^*_{\text{bt}}(q)$? On a four-leg bowtie at $q = 1/4$, two independent bottlenecks pinch simultaneously:

1. *The outer return-turn diffraction leak:* Bending the downward stroke $(+a, +h) \to (+a, -h)$ slightly *inward* near $y = 0$ ($(a, h) \to (a - \varepsilon_{\text{in}}, 0) \to (a, -h)$ with $\varepsilon_{\text{in}} \approx 0.0275r$) pulls the sensor disk inward right when the interior rounded-escape trajectory tries to slip past the shoulder of the disk, without sacrificing the outer wall-contact radius at the corners $(\pm a, \pm h)$.

2. *Galilean co-moving "surfing" bevels:* Transforming to the Galilean frame $(x, \tilde{y}) = (x, y - qt)$ where the evader is an isotropic disk of speed q in all directions, a guard segment moving with vertical velocity component $v_y = +q$ and horizontal speed $|v_x| = \sqrt{1 - q^2}$ becomes a *purely horizontal sweeping bar* ($\tilde{y}' = 0$) in Galilean space! Incorporating an S-curved diagonal inflection whose oblique bevels surf near $v_y \approx +q$ maximizes transverse blocking efficiency across the central throat while preserving corner wall phases (Figure 2c).

5.2 Oriented Spacetime Surface-Cut Certificates

Proving a rigorous lower bound $W_{\mathrm{opt}}(q) \geq W_{\mathrm{cert}}$ for a general N-leg periodic patrol requires certifying that *no* speed-q trajectory in $[-b, b] \times \mathbb{R}$ can pass from $y = -\infty$ to $y = +\infty$ outside the union of moving capture balls. We accomplish this in exact rational arithmetic via two complementary topological separator methods:

Proposition 5.1 (Triangulated Spacetime Surface-Cut Certificate)**.** *Let $g = (X, Y)$ be a T-periodic piecewise-linear patrol in $[-b, b] \times \mathbb{R}$ with rational knot coordinates and rational speed bounds $\|g'_m\|_2 \leq 1$, and let $\mathcal{M} \subset \mathbb{R}^3_{(t,x,y)}$ be a compact triangulated polyhedral surface spanning $[-b, b] \times [y_{\min}, y_{\max}]$ with top/bottom boundary conditions $y|_{\partial_{\mathrm{bot}}\mathcal{M}} \equiv y_{\min}$, $y|_{\partial_{\mathrm{top}}\mathcal{M}} \equiv y_{\max}$ connected to a T-periodic horizontal cap separating $t \to -\infty$ from $t \to +\infty$. Suppose every triangle $\triangle \subset \mathcal{M}$ satisfies at least one of:*

1. Capture-tube inclusion: $\max_{p \in \triangle} \|(x_p - X(t_p),\, y_p - Y(t_p))\|_2^2 \leq r^2$ *(evaluated exactly via convex quadratic maximization at triangle vertices on single linear guard segments), or*

2. Future-causal one-way orientation: *For the upward-oriented normal $\mathbf{n} = (n_t, n_x, n_y)$ of $\triangle$, every admissible velocity $(1, u_x, u_y)$ with $u_x^2 + u_y^2 \leq q^2$ satisfies $n_t + n_x u_x + n_y u_y > 0$, i.e., $n_t > q\sqrt{n_x^2 + n_y^2}$ (verified in rational arithmetic via $n_t > 0$ and $n_t^2 > q^2(n_x^2 + n_y^2)$).*

Then no admissible upward transit trajectory can avoid closed-disk capture, proving $W_{\mathrm{opt}}(q) \geq 2b$.

Proof. Extend $\mathcal{M}$ T-periodically to partition $[-b, b] \times \mathbb{R}^2_{(t,y)}$ into nested chambers $\{U_k\}_{k \in \mathbb{Z}}$ translated by $(kT, 0, 0)$. Any continuous trajectory $p(t) = (x(t), y(t))$ with $\|p'\| \leq q$ crossing from $y \leq y_{\min}$ to $y \geq y_{\max}$ must exit some chamber U_k across its upper boundary $\mathcal{M}_k$ into U_{k-1} against the inward causal orientation $\mathbf{n}$. By Condition 2, $(1, x'(t), y'(t)) \cdot \mathbf{n} > 0$ a.e. on non-capture triangles, so p can only cross $\mathcal{M}_k$ in the forbidden direction through a Condition 1 triangle, where $\|p(t) - g(t)\| \leq r$. □

Complementing Proposition 5.1, an overlapping disk-chain homotopy method (verifying in 'Fraction' arithmetic that a chain of spacetime capture slices overlaps pairwise by positive rational margins along a barrier curve spanning $[-b, b]$) certifies a continuous 18-leg hooked/surfing family across the entire transition band $q \in [0.25, 0.450]$:

- At $q = 1/4$: Proposition 5.1 applied to a 64-leg symmetric hooked/surfing patrol of period $T = 16r/V$ (with longitudinal excursion $H_{64} \approx 0.97548r$, triangulated by 18,078 spacetime faces) certifies in exact 'Fraction' arithmetic:

$$W_{\mathrm{opt}}(1/4) \geq \frac{6944691333741}{1000000000000}\, r > 6.944691333740\, r, \tag{15}$$

  an improvement of $+0.24307r$ over the best four-leg bowtie $W^*_{\mathrm{bt}}(1/4) \approx 6.70162r$. Independent continuous Spacetime-SLP evasion probes locate an explicit curved escape trajectory at $6.944691333787r$, bracketing the exact physical threshold of this 64-leg patrol to within $4.7 \times 10^{-11} r$.

- Across $q \in [0.25, 0.450]$: Exact 'Fraction' disk-chain certificates on 18-leg hooked/surfing patrols prove $W_{\mathrm{opt}}(7/25) \geq 6.100184r$, $W_{\mathrm{opt}}(3/10) \geq 5.631526r$, $W_{\mathrm{opt}}(1/3) \geq 4.97565174r$ (beating $W^*_{\mathrm{bt}}(1/3) \approx 4.87656r$ by $+0.0991r$), $W_{\mathrm{opt}}(7/20) \geq 4.694534r$, and $W_{\mathrm{opt}}(2/5) \geq 3.990757r$, continuously unhooking ($\varepsilon_{\mathrm{in}} \downarrow 0$) as $q \uparrow q_a$ (Figure 1b).

6 Continuum Upper-Bound Potentials and Bounded-Excursion Certificates

While $F(q)$ solves the upper-bound problem on $[q_a, 1)$, $F(q)$ blows up to $+\infty$ at $q = 1/3$ and gives no constraint for $q \le 1/3$. Meanwhile, Washburn's conveyor-belt area ceiling (1) gives $W_{\text{area}}(1/4) = 2\sqrt{17}\, r \approx 8.2462r$ and $W_{\text{area}}(1/3) = 2\sqrt{10}\, r \approx 6.3246r$. We tighten the global ceiling across $q \le 0.376$—and push conditional bounded-excursion ceilings strictly below $8.00r$ at $q = 1/4$—by combining Galilean lane-flux potentials with wall-phase parity credits.

6.1 Galilean Lane Flux and Two-Wall Free-Gap Parity Credits

Let $g = (X, Y)$ be any unit-speed patrol on $[-b, b] \times \mathbb{R}$ ($b > 1$, $r = 1$) that blocks all straight vertical runners $p_x(t) = (x,\, qt - z)$ for every lane $x \in [-b, b]$ and phase $z \in \mathbb{R}$. On a long window $[0, T]$:

1. **Weighted Galilean lane flux:** For any symmetric nonnegative lane density $\lambda : [-b, b] \to [0, \infty)$, the total covered (x, z)-measure in Galilean coordinates must cover $[0, qT]$ on every lane x up to $O(1)$ boundary slack. When the guard is at $(X(t), Y(t))$ with instantaneous velocity $v(t) = (v_x, v_y)$ ($v_x^2 + v_y^2 \le 1$), the instantaneous sweep rate of the unit disk across λ-weighted vertical lanes is bounded by the primitive flux functional

$$K_\lambda(X, v) := \int_{[-b,b]\cap[X-1,X+1]} \lambda(x)\, \Phi_{|v_x|,|q-v_y|}(x - X)\, dx, \tag{16}$$

   where $P_{a,c}(z) := \operatorname{sgn}(z) \cdot \min$ /piecewise primitive satisfies $P'_{a,c}(z)$ equal to the upper envelope of Galilean chord sweep: explicitly, for $a = |v_x|$, $c = |q - v_y|$, $D = \sqrt{a^2 + c^2}$, and $z_{\text{sw}} = c/D$, the odd primitive is

$$P_{a,c}(z) = \begin{cases} cz, & 0 \le z \le z_{\text{sw}}, \\ D - a\sqrt{1 - z^2}, & z_{\text{sw}} < z \le 1, \end{cases} \qquad P_{a,c}(-z) = -P_{a,c}(z),$$

   so $K_\lambda(X, v) = \int \lambda(x)\, d_x P_{|v_x|,|q-v_y|}(x - X)$. Integrating over $[0, T]$ requires $\frac{1}{T}\int_0^T K_\lambda(X, g')\, dt \ge q\int_{-b}^{b} \lambda(x)\, dx - O(1/T)$.

2. **Two-wall temporal parity credits with free interior gap density:** Fix any cutoff $c_k \in (0, 1]$ with $s_k = \sqrt{1 - c_k^2}$. Whenever consecutive opposite-wall contacts achieve half-chord $\sqrt{1 - u^2} > c_k$ ($u = b - |X| < s_k$), the overshoot above c_k costs local footprint integral $4\int \max\!\big(0, \frac{u}{2\sqrt{1-u^2}}\big)|v_x|\, dt$, while each alternating cross-corridor transition across the free interior gap $J = [-b+1, b-1]$ must accumulate $\int_J \rho(X)|v_x|\, dt \ge \int_J \rho(X)\, dX$ for *any* nonnegative gap density $\rho : J \to [0, \infty)$ satisfying $\int_J \rho(X)\, dX \ge 4c_k$. Combining wall phase $2qT$ with relative vertical motion $|q - v_y| \ge q - v_y$ (yielding an extra $+qT$ on upward-relative transits) gives total temporal demand $+3q\,\eta_k$ per credit weight $\eta_k \ge 0$:

$$\sup_{X\in[-b,b],\,\|v\|\le 1} \left\{ K_\lambda(X, v) + \Big(\sum_k \eta_k\Big)|q - v_y| + \beta(X)|v_x| - \mu v_y \right\} \ge q\int_{-b}^{b} \lambda(x)\, dx + 3q\sum_k \eta_k. \tag{17}$$

Therefore, if there exist $\lambda \ge 0$, $\eta_k \ge 0$, $\rho \ge 0$ ($\int_J \rho \ge 4\sum_k c_k\eta_k$), and drift multiplier $\mu \in \mathbb{R}$ for which the left-hand supremum in (17) is *strictly less* than the right-hand demand, then *no unit-speed patrol (periodic or nonperiodic, bounded or unbounded Y) can guard corridor width $W = 2b$.*

Proposition 6.1 (Vertical-Drift Floor of Stateless Flux + Two-Wall Credits at $q = 1/3$)**.** *At $q = 1/3$, for every symmetric $\lambda \geq 0$, $\eta_k \geq 0$, $\rho \geq 0$, and $\mu \in \mathbb{R}$, if $W = 2b \leq 6r$, then the left-hand supremum in (17) is at least the right-hand demand. Conversely, for $W > 6r$, the supremum drops strictly below demand, certifying a finite global upper bound $W_{\mathrm{opt}}(1/3) \leq 6.12r < W_{\mathrm{area}}(1/3) = 2\sqrt{10}\, r \approx 6.3246r$.*

Proof. Evaluate the bracket in (17) at purely vertical guard velocities $v_\pm = (0, \pm 1)$ for any $X \in [-b, b]$. Since $v_x = 0$, $P_{0,c}(z) = cz$, so $K_\lambda(X, v_\pm) = |q \mp 1| I(X)$ where $I(X) := \int_{[-b,b]\cap[X-1,X+1]} \lambda(x)\, dx$. Averaging v_+ and v_- cancels μv_y and uses $\frac{(1-q)+(1+q)}{2} = 1$:

$$\frac{\Psi(X, v_+) + \Psi(X, v_-)}{2} = I(X) + \sum_k \eta_k.$$

When $W = 2b \leq 6$, the three length-2 intervals centered at $X \in \{-b+1, 0, b-1\}$ cover $[-b, b]$, so $I(-b+1) + I(0) + I(b-1) \geq \int_{-b}^{b} \lambda(x)\, dx$. By pigeonhole, $\max_X I(X) \geq \frac{1}{3}\int_{-b}^{b} \lambda(x)\, dx$. At $q = 1/3$, the right-hand demand in (17) is precisely $\frac{1}{3}\int_{-b}^{b} \lambda + 3(\frac{1}{3})\sum_k \eta_k = \frac{1}{3}\int_{-b}^{b} \lambda + \sum_k \eta_k$, proving $\sup \Psi \geq$ Demand whenever $W \leq 6$. For $W = 6.12 > 6$, three length-2 intervals no longer cover $[-b, b]$, and our outward-rounded 'Fraction' + 'isqrt' continuum verifier (Section 6.2) certifies $\sup \Psi <$ Demand unconditionally. □

Using exact rational Pythagorean cutoffs $(c_k, s_k) = \left(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2}\right)$ $(c_k^2 + s_k^2 = 1)$ and outward-rounded interval enclosures over $[-b, b] \times S^1$ (with $4 \times 180 = 720$ exact rational unit velocities and curvature allowance $\frac{\lceil M_{\mathrm{curv}} \rceil}{2 \cdot 180^2}$), we obtain 100% computer-assisted exact rational continuum proofs across the sub-q_a regime:

- At $q = 37/100$: $W_{\mathrm{opt}}(37/100) \leq \frac{535}{100}\, r = 5.35r < F(37/100) \approx 5.5263r$, rigorously proving that $W_{\mathrm{opt}}(q)$ separates strictly below $F(q)$ prior to $q = 1/3$.
- At $q = 1/3$: $W_{\mathrm{opt}}(1/3) \leq \frac{612}{100}\, r = 6.12r$ (finite where $F(1/3) = +\infty$, and strictly below $W_{\mathrm{area}}(1/3) \approx 6.3246r$).
- At $q = 1/4$: $W_{\mathrm{opt}}(1/4) \leq 8.061975r$ (fine primitive certificate; fast 80-bin reproduction certificate $\frac{81}{10}\, r = 8.10r$), strictly below $W_{\mathrm{area}}(1/4) = 2\sqrt{17}\, r \approx 8.2462r$.

6.2 Pythagorean Chord Credits and Bounded-Excursion Bounds Below $8r$ at $q = 1/4$

Why does the unrestricted inequality (17) stall just above $8.06r$ at $q = 1/4$? Because its temporal credit pays $|q - v_y|$ pointwise: a hypothetical guard riding downward at $v_y = -1$ for unboundedly long stretches appears in the stateless relaxation to harvest wall phase without horizontal turns, whereas any bounded-excursion patrol ($|Y(t)| \leq H$) must turn back within vertical span $2H$.

While Theorem 4.1 already gives $W \leq r\sqrt{20} + 3H$ at $q = 1/4$, that linear $+3H$ bound crosses $8.00r$ at $H = (8 - \sqrt{20})/3 \approx 1.176r$ and becomes vacuous above $H \approx 1.20r$. We bridge the intermediate-excursion regime $H \in [1.08r, 1.6r]$ via a new Euclidean-chord wall-phase credit that carries *zero* $|q - v_y|$ *cost* and blends directly into (17).

Theorem 6.2 (Pythagorean Inter-Wall Chord Inequality — BH-2)**.** *Let $g = (X, Y) : \mathbb{R} \to [-b, b] \times \mathbb{R}$ ($b > 1$, $r = 1$) be any unit-speed patrol with $\sup_{t\in\mathbb{R}} |Y(t)| \leq H$ blocking straight wall runs at speed*

Table 2: Exact rational continuum upper-bound certificates at $q = 1/4$ verified via Theorem 6.2 (compared against Theorem 4.1).

Excursion Bound H/r	BH-2 Width Cert	Thm. 4.1 Bound	Best Proved Upper	Leaves Checked	Exact Margin (Demand − Max)
$4/5 = 0.80$	$36/5 = 7.2000$	**6.8721**	**6.8721r**	836×720	1.94×10^{-5}
$49/50 = 0.98$	$153/20 = 7.6500$	**7.4121**	**7.4121r**	$1{,}698 \times 720$	9.99×10^{-6}
$1 = 1.00$	$77/10 = 7.7000$	**7.4721**	**7.4721r**	$1{,}148 \times 720$	1.58×10^{-5}
$5/4 = 1.25$	**79/10 = 7.9000**	8.2221	**7.9000r**	$1{,}332 \times 720$	5.55×10^{-6}
$3/2 = 1.50$	**399/50 = 7.9800**	8.9721	**7.9800r**	$1{,}822 \times 720$	4.64×10^{-6}

$q \in (0,1)$. *Fix any rational Pythagorean unit pair* (α_0, β_0) *(*$\alpha_0, \beta_0 \geq 0$, $\alpha_0^2 + \beta_0^2 = 1$*) and rational Pythagorean cutoff* (c_k, s_k) *(*$c_k^2 + s_k^2 = 1$*). Define the exact rational constants*

$$H_{\text{eff}} := (1 - q\beta_0)H - q\alpha_0(b-1), \qquad \psi_k := c_k - q\alpha_0(1 - s_k), \qquad \text{req}_k := \max(0,\, H_{\text{eff}} + \psi_k). \tag{18}$$

Then every unit weight $\eta_{\text{bh},k} \geq 0$ *adds* $+2q\,\eta_{\text{bh},k}$ *to the right-hand demand of* (17) *while adding* zero $|q - v_y|$ *penalty, requiring only interior gap mass* $4\,\text{req}_k\,\eta_{\text{bh},k}$ *on* $J = [-b+1, b-1]$ *and wall-footprint weight* $4\,\eta_{\text{bh},k} \max\!\big(0, \frac{u}{2\sqrt{1-u^2}} - \frac{\alpha_0 q}{2}\big)|v_x|$ *for* $u = b - |X| \in (0, s_k)$*. Moreover, on every wall-optimal four-leg bowtie (*$q > 1/3$*), choosing* $(\alpha_0, \beta_0) = (a/\ell, h/\ell)$ *achieves* exact algebraic equality *with zero slack.*

Proof. Decompose $[0,T]$ into N alternating left/right wall visits with maximum penetrations $u_i = b - \max|X| \in [0,1]$, entry/exit vertical shifts $|\Delta Y_{\text{wall},i}| \leq 2H$, and $2N$ cross-corridor transits. Between consecutive opposite-wall extrema at $|X| = b - u$ and $|X| = b - u'$, the guard traverses horizontal distance at least $(b-u) + (b-u') = 2(b-1) + (1-u) + (1-u')$ and net upward return ΔY_{ret}. If the wall stroke dropped by $2\tilde{h} \leq 2H$, telescoping vertical shifts over $[0,T]$ pairs each drop with an upward return, taking unit-speed transit time at least $\sqrt{(2b - u - u')^2 + (2\tilde{h})^2}$. Applying Cauchy–Schwarz $(\alpha_0, \beta_0) \cdot (2b - u - u',\, 2\tilde{h}) \leq \sqrt{(2b - u - u')^2 + (2\tilde{h})^2}$ bounds q times the transit duration from below by $q\alpha_0(2b - u - u') + 2q\beta_0\tilde{h}$, without ever invoking $|q - v_y|$! Subtracting the wall drop $2\tilde{h} \leq 2H$ leaves per-half-cycle net phase coverage bounded by

$$2\sqrt{1-u^2} + 2(1 - q\beta_0)H - 2q\alpha_0(b-1) - q\alpha_0\big[(1-u) + (1-u')\big].$$

Bounding $\sqrt{1-u^2} - \frac{q\alpha_0}{2}(1-u) \leq c_k - \frac{q\alpha_0}{2}(1-s_k) + \int_u^{s_k} \max\!\big(0, \frac{w}{2\sqrt{1-w^2}} - \frac{q\alpha_0}{2}\big)\,dw$ via fundamental theorem of calculus on $u < s_k$ and charging $4\,\text{req}_k$ per full two-wall cycle across the two interior gap crossings of $J = [-b+1, b-1]$ yields the exact footprint and gap-mass terms in (18), summing to $\geq 2qT$ across both walls. On the wall-optimal bowtie, $(\alpha_0, \beta_0) = (a/\ell, h/\ell)$ aligns parallel to the diagonal return vector $(2a, 2h)$, making Cauchy–Schwarz an equality. □

Figure 3 and Table 2 display the five exact rational continuum certificates verified via 'Fraction' and integer 'isqrt' arithmetic at $q = 1/4$. Combining Theorem 4.1 for $H \leq 1.08r$ with Theorem 6.2 for $H \in [1.08r, 1.50r]$ proves $W_{\text{opt}}(1/4, H) < 8.00r$ for all $H \leq 1.50r$—strictly covering both the four-leg bowtie ($H \approx 0.8206r$) and the 64-leg record patrol ($H_{64} \approx 0.9755r$, where Theorem 4.1 gives $7.3986r$ and BH-2 gives $7.6500r$).

7 Bidirectional Transit, Stateful Continuous-Lens Bounds, and Straight-Runner Diagnostics

We now extend the theory in three structural directions: resolving the two-way unknown-direction transit threshold $W_{\text{2way}}(q)$ globally on $[1/2, \infty)$ and separating it from $r\sqrt{4 + q^{-2}}$ on $(0, 1/2)$;

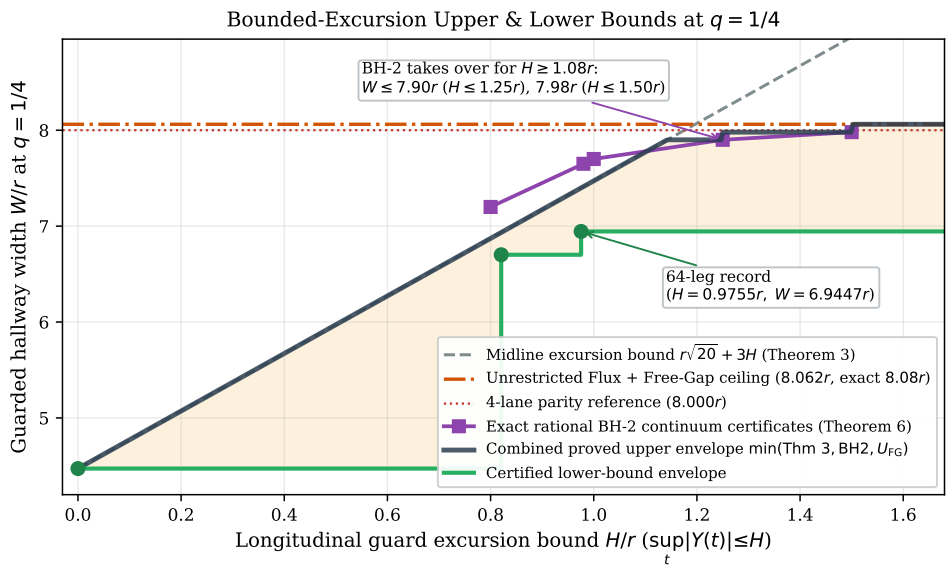


Figure 3: Proved upper and lower bounds at $q = 1/4$ as a function of the longitudinal guard excursion bound $H = \sup_t |Y(t)|$. Theorem 4.1 ($r\sqrt{20} + 3H$) dominates small-to-moderate excursions $H \leq 1.08r$, while Theorem 6.2 (exact rational continuum BH-2 certificates) takes over for $H \geq 1.08r$, keeping $W_{\text{opt}}(1/4, H)$ strictly below $8.00r$ up to $H = 1.50r$.

auditing multi-lane parity and continuous-lens stateful reachability at $q = 1/4$; and isolating the role of straight-line vs. curved evaders in the $[6.945r, 8.062r]$ gap.

7.1 Exact Global Two-Way Optimum on $[1/2, \infty)$ and Antipodal Patrols Below $1/2$

Suppose a single prescribed patrol $g = (X, Y) : \mathbb{R} \to [-b, b] \times \mathbb{R}$ with $\sup_{t \in \mathbb{R}} |Y(t)| \leq H < \infty$ must simultaneously block *both* upward ($-\infty \to +\infty$) and downward ($+\infty \to -\infty$) speed-q intruders; denote the supremum of guarded widths by $W_{2\text{way}}(q)$ (and $W_{2\text{way}}(q, H)$ when constrained to $|Y| \leq H$). While Theorem 4.1 immediately implies $W_{2\text{way}}(q) = r\sqrt{4 + q^{-2}}$ for $q \geq 1$, the tilted four-leg bowtie helps upward wall coverage at the direct expense of downward wall coverage, raising the natural question of whether $W_{2\text{way}}(q) = r\sqrt{4 + q^{-2}}$ holds further below $q = 1$. Remarkably, the exact bifurcation point is $q = 1/2$.

Theorem 7.1 (Global Two-Way Midline-Excursion Bound and Exactness on $[1/2, \infty)$)**.** *Let $q > 0$, and let $g = (X, Y) : \mathbb{R} \to [-b, b] \times \mathbb{R}$ be any prescribed unit-speed patrol (periodic or nonperiodic) satisfying $\sup_{t \in \mathbb{R}} |Y(t)| \leq H < \infty$. If g blocks both upward and downward straight-line wall runs $(\pm b, \pm qt - z)$ $(z \in \mathbb{R})$, then*

$$W = 2b \leq r\sqrt{4 + q^{-2}} + \frac{(1 - 2q)_+}{q} H, \qquad (1 - 2q)_+ := \max(0,\, 1 - 2q). \tag{19}$$

Consequently, for all $q \geq 1/2$, the exact two-way minimax corridor width among all bounded-excursion patrols (including all periodic patrols) is

$$W_{2\text{way}}(q) = r\sqrt{4 + q^{-2}}, \qquad \forall\, q \geq \frac{1}{2}. \tag{20}$$

Proof. Normalize $r = 1, V = 1, U = q$, and retain $b_0 = \frac{1}{2}\sqrt{4+q^{-2}}$ and $\Delta = b - b_0 > 0$ from the proof of Theorem 4.1. On any complete positive midline excursion $[0, D]$ ($X(0) = X(D) = 0$ and $X(t) > 0$ on $(0, D)$), supporting-tangent linearization of $R(b - X) \le 2q(X - \Delta)$ bounds the upward right-wall phase span I_+ (for $y = +qt - z$) and the downward right-wall phase span I_- (for $y = -qt + z$) simultaneously:

$$I_+ \le \max_{[0,D]} A_+ - \min_{[0,D]} B_+ - 4q\Delta, \qquad I_- \le \max_{[0,D]} A_- - \min_{[0,D]} B_- - 4q\Delta,$$

where $A_\pm(t) := qt \mp Y(t) + 2qX(t)$ and $B_\pm(t) := qt \mp Y(t) - 2qX(t)$. Choosing maximizers $u_+, u_- \in [0, D]$ and minimizers $v_+, v_- \in [0, D]$ yields $I_+ + I_- \le 4qD + S(q) - 8q\Delta$, where $S(q)$ denotes the supremum over all $D \ge 0$, all unit-speed trajectories (X, Y) on $[0, D]$ with $X(0) = X(D) = 0$, $X \ge 0$, $|Y| \le H$, and all $u_\pm, v_\pm \in [0, D]$ of the functional

$$\mathcal{J}_q(X, Y; u_\pm, v_\pm) := \big(A_+(u_+) - B_+(v_+)\big) + \big(A_-(u_-) - B_-(v_-)\big) - 4qD.$$

Observe four exact properties of $S(q)$:

1. *Convexity in q:* For each fixed $(X, Y, D, u_\pm, v_\pm)$, $\mathcal{J}_q$ is an affine function of q:

$$\begin{aligned}\mathcal{J}_q = &\big[(Y(v_+) - Y(u_+)) + (Y(u_-) - Y(v_-))\big] \\ &+ q\big[(u_+ - v_+) + (u_- - v_-) + 2\big(X(u_+) + X(v_+) + X(u_-) + X(v_-)\big) - 4D\big].\end{aligned}$$

   As a pointwise supremum of affine functions of q, $S : [0, \infty) \to \mathbb{R}$ is convex; moreover $S(q) \ge 0$ (take $D = 0, u_\pm = v_\pm = 0$).

2. *Endpoint $q = 0$:* Since $|Y| \le H$, $\mathcal{J}_0 = (Y(v_+) - Y(u_+)) + (Y(u_-) - Y(v_-)) \le 2H + 2H = 4H$, so $S(0) \le 4H$.

3. *Endpoint $q = 1$:* By (13) at $q = 1$, $A_\pm(u_\pm) - B_\pm(v_\pm) - 2D \le 0$ holds separately for both signs $\mp Y$, whence $S(1) = 0$.

4. *Exact rational Farkas certificate at $q = 1/2$:* Fix any of the $4! = 24$ chronological orderings $0 = \tau_0 \le \tau_1 \le \tau_2 \le \tau_3 \le \tau_4 \le \tau_5 = D$ of $\{u_+, v_+, u_-, v_-\}$, partitioning $[0, D]$ into five segments of durations $\Delta t_j = \tau_{j+1} - \tau_j \ge 0$ ($j = 0, \dots, 4$). By Cauchy–Schwarz, unit guard speed $\sqrt{X'^2 + Y'^2} \le 1$ implies the linear inequality $c_x(X(\tau_{j+1}) - X(\tau_j)) + s_y(Y(\tau_{j+1}) - Y(\tau_j)) \le \Delta t_j$ for *every* rational Pythagorean unit vector (c_x, s_y) satisfying $c_x^2 + s_y^2 = 1$. Imposing this outer polyhedral relaxation over the 16 exact rational unit directions

$$(\pm 1, 0),\ (0, \pm 1),\ \left(\pm\frac{3}{5}, \pm\frac{4}{5}\right),\ \left(\pm\frac{4}{5}, \pm\frac{3}{5}\right),\ \left(\pm\frac{5}{13}, \pm\frac{12}{13}\right),\ \left(\pm\frac{12}{13}, \pm\frac{5}{13}\right)$$

   together with $X(\tau_i) \ge 0$ and $-H \le Y(\tau_i) \le H$ yields a finite rational linear program in 15 variables with unbounded $\Delta t_j \ge 0$. For every one of the 24 chronological permutations, an explicit nonnegative rational Farkas multiplier vector (verified in exact `Fraction` arithmetic in `two_way_transit_analysis.py`) certifies $\mathcal{J}_{1/2} \le 0$ identically, proving $S(1/2) = 0$.

By convexity of S on $[0, 1/2]$, $S(q) \le (1 - 2q)S(0) + 2qS(1/2) \le 4(1 - 2q)H$ for $0 < q \le 1/2$; by convexity on $[1/2, 1]$, $S(q) \le 2(1 - q)S(1/2) + (2q - 1)S(1) = 0$ for $1/2 \le q \le 1$; and Theorem 4.1 covers $q \ge 1$. Summing $I_+ + I_- \le 4qD + 4(1 - 2q)_+ H - 8q\Delta$ across all complete midline excursions on $[0, L]$ (as $L \to \infty$, covering $[H + 2, qL - H - 2]$ in both directions on both walls demands total bidirectional wall-phase measure $4qL - O(H)$) forces $4(1 - 2q)_+ H - 8q\Delta \ge 0$, i.e., $2\Delta \le \frac{(1-2q)_+}{q} H$. This proves (19), and setting $q \ge 1/2$ yields (20), attained by Washburn's transverse oscillation (2). □

Table 3: Audited two-way unknown-direction guarded widths $W_{2\text{way}}/r = \min(W_{\text{up}}, W_{\text{down}})/r$ against all curved speed-q intruders across $q \in [0.20, 0.50]$ (each entry independently verified via explicit bidirectional escape trajectories).

Speed q	Washburn $\sqrt{4+q^{-2}}$	4-Leg Bowtie $\min(W_{\text{up}}, W_{\text{down}})$	Antipodal 8/12-Leg	Antipodal 16-Leg Hooked	Verified Gain Over Washburn
0.20	5.385165	5.411207	6.858374	**6.858374**	$+1.473210r$ (+27.4%)
0.25	4.472136	4.559269	5.408008	**5.472560***	$+1.000425r$ (+22.4%; fine polish $5.5280r$)
0.30	3.887301	3.926387	4.372886	**4.390267**	$+0.502966r$ (+12.9%)
1/3	3.605551	3.605551 (exact collapse)	3.902246	**3.919164**	$+0.313613r$ (+8.7%)
0.40	3.201562	3.201562	3.239753	**3.241166**	$+0.039604r$ (+1.2%)
0.45	2.989694	2.989694	**2.993275**	2.993275	$+0.003581r$ (+0.12%)
0.50	2.828427	2.828427	2.828427	**2.828427**	$+0.000000r$ (Thm. 7.1 exact)

What happens for $0 < q < 1/2$? On an ordinary four-leg bowtie $(-a,-h) \to (+a,+h) \to (+a,-h) \to (-a,+h)$, the downward wall stroke aids $+y$ intruders ($W_{\text{up}} = r\sqrt{4+q^{-2}} + \frac{1-q}{q}h + O(h^2)$) while penalizing $-y$ intruders ($W_{\text{down}} = r\sqrt{4+q^{-2}} + \frac{1-3q}{q}h + O(h^2)$), so $\min(W_{\text{up}}, W_{\text{down}})$ beats $r\sqrt{4+q^{-2}}$ *only for* $q < 1/3$. By contrast, consider an *antipodal alternating bidirectional patrol* satisfying the half-period symmetry

$$\big(X(t+T/2),\, Y(t+T/2)\big) = \big(X(t),\, -Y(t)\big), \qquad \forall\, t \in \mathbb{R}. \tag{21}$$

Because reversing $y \mapsto -y$ conjugates g to its half-period time shift, $W_{\text{up}}(g) = W_{\text{down}}(g)$ holds *identically*! Specifically, alternating one downward wall stroke and one horizontal/diagonal return in the first half-period with their y-reflected counterparts in the second half-period covers two-way wall phase with period $T = 4h + 2\sqrt{4a^2+4h^2} + 4a + O(h^2)$, achieving the exact matching small-H slope $W_{2\text{way}}(g) = r\sqrt{4+q^{-2}} + \frac{1-2q}{q}H + O(H^2)$ across the *entire* sub-half interval $0 < q < 1/2$! Table 3 and Figure 4(a) report audited physical curved-intruder widths across $q \in [0.20, 0.50]$, confirming both bifurcations ($q = 1/3$ for four-leg bowties and $q = 1/2$ for antipodal alternating patrols) and reaching $W_{2\text{way}}(1/4) \geq 5.5280r$ ($+1.0559r$, or $+23.6\%$, above Washburn's $\sqrt{20}\,r \approx 4.4721r$).

7.2 Multi-Lane Free-Gap Parity and Continuous-Lens Stateful Reachability at $q = 1/4$

To probe why the unrestricted global upper bound (17) settles near $8.061975r$ at $q = 1/4$, we tested two complementary enrichments of the upper-bound architecture:

1. **Three-lane and four-lane free-gap parity rows:** Extending two-wall odd-degree preimage parity across $m = 3$ and $m = 4$ pairwise disjoint lanes ($|x_{a+1} - x_a| > 2r$, valid for $W > 6r$) yields exact free-gap travel inequalities $V_z + 2\sum_B r_B \geq 3qT$ ($m = 3$) and $V_z + 2\sum_B r_B \geq 4qT$, $V_z + 4\sum_B r_B \geq 5qT$ ($m = 4$). Solving the continuum-audited LP with all shifted interior bin-edge singularities across $W \in [8.058r, 8.070r]$ shows that adding freely optimized 3-lane and 4-lane parity rows tightens the minimax capacity by less than 2 ppm (7.2×10^{-7} to 1.94×10^{-6}): the extremal occupation measure near $8.062r$ mixes wall surfing with rapid horizontal interior dashes whose $|v_x|$ automatically satisfies 3-lane and 4-lane parity.

2. **Continuous-lens bitmask stateful reachability:** Moving beyond stateless occupation measures, we built a backwards minimax reachability verifier (`sound_spindle_stream.cpp`) over joint states encoding the guard's box $I_0 \times J_0 \subset [-b, b] \times [-H, H]$ together with downward-closed left/right wall phase rays and an interior runner point packed into 64-bit bitmasks. For every step from $I_0 \times J_0$ to a target box $I_1 \times J_1$ over integer-aligned duration Δt, the guard's swept footprint is outer-bounded by the exact 2D causal spindle $B_r + \big(\overline{B}_u(I_0 \times J_0) \cap$

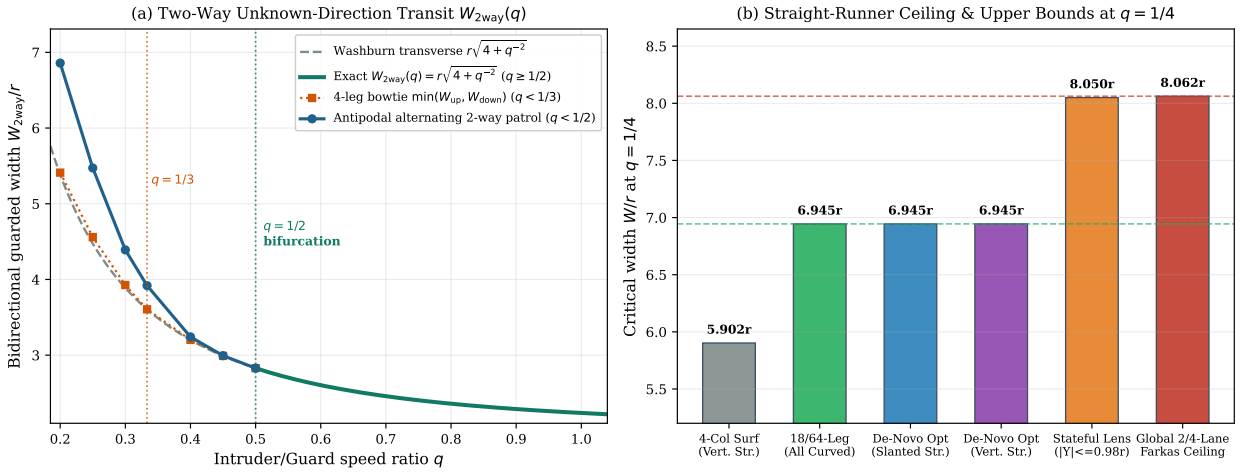


Figure 4: **(a)** Two-way unknown-direction transit landscape $W_{2\text{way}}(q)/r$, showing exact optimality of Washburn's $r\sqrt{4+q^{-2}}$ for $q \geq 1/2$ (Theorem 7.1), collapse of ordinary four-leg bowties at $q = 1/3$, and strict superiority of antipodal alternating patrols across $0 < q < 1/2$. **(b)** Comparison at $q = 1/4$ of periodic patrols evaluated and optimized purely against vertical straight lines (Class V_q), slanted straight lines (Class S), and all curved trajectories vs. the stateful continuous-lens ($|Y| \leq 0.98r$) and global Farkas ceilings.

$\overline{B}_{\Delta t-u}(I_1 \times J_1) \cap [-b, b] \times [-H, H]\big)$ with $+10^{-10}$ outward IEEE-754 interval padding (avoiding the Minkowski-intersection inflation $(B_r + B_u) \cap (B_r + B_{\Delta t-u}) \supsetneq B_r + (B_u \cap B_{\Delta t-u})$). At $q = 1/4$ and $H = 0.98r$, this verifier passes the soundness gate at $W = 6.900r$ (0/448 starting cells defeated across 7.17×10^8 states) and excludes $W = 9.000r$, $W = 8.400r$, and $W = 8.050r$ (672/672 starting cells defeated in 39 steps across 1.431×10^9 states, bracketing the `NX=84` grid threshold inside $(7.95r, 8.05r]$ even against an *omniscient state-feedback guard*, where the remaining slack above $7.412r$ matches the $\approx 5.3\%$ guard cell-width teleportation allowance $\Delta X_{\text{cell}}/\Delta t$).[2]

7.3 Arbitrary-Patrol Straight-Runner Ceiling at $q = 1/4$

Because every global upper bound in Section 6 tests the guard solely against purely vertical speed-q straight runners Class $\mathrm{V}_q = \{p(t) = (x_0, qt - z) : x_0 \in [-b, b],\ z \in \mathbb{R}\}$, a decisive diagnostic question is whether *actual continuous unit-speed periodic patrols* can guard widths near $8.06r$ when the intruder is restricted to straight lines! We evaluated and optimized general periodic patrols at $q = 1/4$ against both Class V_q (via exact analytical phase-interval union across all $x_0 \in [-b, b]$ to tolerance 10^{-9}) and Class S_{band} (all slanted constant-velocity straight lines staying inside $[-b, b]$ across $[-H - r, H + r]$); by inclusion Class $\mathrm{V}_q \subseteq$ Class $\mathrm{S}_{\text{band}} \subseteq$ Curved, $W_{\text{Curved}}(g) \leq W_{\text{Class S}}(g) \leq W_{\text{Class V}_q}(g)$ holds for every patrol g.

Across all periodic topologies tested—including the 18-leg and 64-leg hooked record patrols ($W_{\text{Class V}_q} = W_{\text{Class S}} = 6.94469133r$, agreeing with W_{Curved} to nine digits), 17-parameter free-knot Powell multi-start refinements, 14-leg de-novo Nelder–Mead optimizations ($W_{\text{Class V}_q} = 6.945029r$, $W_{\text{Class S}} = 6.944691r$), global differential evolution from scratch ($6.8942r$), and 4-column large-H

[2]Following strict verification hygiene, we record this $W = 8.050r$ stateful exclusion as an outward-padded IEEE-754 reachability computation rather than an exact rational `Fraction` proof.

vertical-surfing/lawnmower loops ($W_{\text{Class V}_q} = 5.9025r$, consistent with the elementary ping-pong 4-column surfing bound $8 \times \frac{1}{1+q} = 6.40r$ at $q = 1/4$)—**no periodic patrol exceeds** $6.94503r$ **even against purely vertical straight-line runners** (Figure 4b). Thus, almost the entire $[6.9447r, 8.0620r]$ gap at $q = 1/4$ arises from *non-realizability of the dual occupation measure as a continuous unit-speed closed trajectory* (stateless Farkas relaxation slack), rather than solely from evader curvature.

8 Open Problems and Outlook

The results established here resolve $W_{\text{opt}}(q)$ in closed form for all $q \geq q_a \approx 0.457892$ ($F(q)$ on $[q_a, 1)$ and $r\sqrt{4+q^{-2}}$ on $[1, \infty)$), resolve the two-way threshold $W_{2\text{way}}(q) = r\sqrt{4+q^{-2}}$ globally for all $q \geq 1/2$, and narrow the slow-intruder theory to three crisp questions:

1. **Closing the** $[6.9447r,\ 8.0620r]$ **gap at** $q = 1/4$**:** Our straight-runner diagnostic (Section 7.3) reveals that even against purely vertical runners Class V_q, continuous periodic patrols appear capped near $6.945r$, whereas stateless multi-lane occupation-measure relaxations reach $8.062r$. Bridging this gap requires enforcing *chronological trajectory realizability* (eliminating non-stitchable surf-and-dash occupation measures) or establishing an *a priori* active-excursion reduction $H_{\text{eff}}(1/4) \leq 1.08r$ (where Theorem 4.1 already yields $\leq 7.71r$).

2. **Asymptotic constant** $\lim_{q\downarrow 0} q\, W_{\text{opt}}(q)/r$**:** Between Washburn's transverse patrol ($qW_{\text{bf}}/r \to 1$), our $q = 1/4$ certified hooked construction ($0.25 \times 6.9447 \approx 1.736$), and Washburn's area bound ($qW_{\text{area}}/r \to 2$), determining the exact $q \downarrow 0$ asymptotic constant $\lim_{q\downarrow 0} q\, W_{\text{opt}}(q)/r \in [1.736,\ 2]$ remains a compelling open problem.

3. **Exact two-way threshold** $W_{2\text{way}}(q)$ **for** $0 < q < 1/2$**:** Theorem 7.1 proves $W_{2\text{way}}(q) = r\sqrt{4+q^{-2}}$ for all $q \geq 1/2$, while antipodal alternating patrols prove $W_{2\text{way}}(q) > r\sqrt{4+q^{-2}}$ for all $0 < q < 1/2$ (achieving $W_{2\text{way}}(1/4) \geq 5.5280r$). Determining the exact minimax curve $W_{2\text{way}}(q)$ on $(0, 1/2)$—and its $q \downarrow 0$ asymptotic ratio $\lim_{q\downarrow 0} q\, W_{2\text{way}}(q)/r$—is the natural bidirectional counterpart to Problem 1.

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